THE ELASTIC INSTABILITY OF DURALUMIN COLUMNS IN COMPRESSION

by

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A. B., Harvard University, 1932

Submitted in Partial Fulfillment of the Requirements

for the degree of

MASTER OF SCIENCE

from the

Massachusetts Institute of Technology

1933

Signature of Author..

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Cambridge, Mass.
May 25, 1933

Professor A. L. Merrill
Secretary of the Faculty
Massachusetts Institute of Technology
Cambridge, Mass.

Dear Sir:

I am submitting herewith my thesis, "The Elastic Instability of Duralumin Columns in Compression" as partial fulfillment of the requirements for the degree of Master of Science.

Very respectfully yours,

Archibald B. Callender

ACKNOWLEDGMENT

The author wishes to express his appreciation for the advice and assistance given him by Professor J. S. Newell of the Aeronautical Engineering Department, under whose direction this work was carried out

Thanks are also due to the National Advisory Committee for Aeronautics; the Information Division, Air Corps; and the Aeronautics Bureau, Navy Department, which supplied most of the test data used in this thesis.

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OBJECT

The object of this thesis was to apply mathematical formulas based on elastic theory to duralumin columns in an attempt to predict the failing loads in the range below that characterized by Euler failures.

The results of the investigation showed that if certain relations in the dimensions of the columns were fulfilled, the theory gave a good approximation to the critical stress.

INTRODUCTION

The practical aspect of the problem of instability of compression members appears in connection with the buckling of the outstanding flanges of various sections subject to end loads.

It has been observed for a number of years that the tendency of design towards greater strength by distributing material at a distance from the neutral axis in the form of wide and thin flanges, or other parts, has been limited by the possibility of a type of secondary failure of the part itself, rather than the member as a whole. Compression members having angle, or channel, or similar sections, in which the outstanding legs are wide and thin, or in which the whole section is made up of thin parts, show a tendency to fail through local wrinkling or twisting, rather than the ordinary failures of direct compression or bending. Local buckling of an outstanding leg may occur, or the whole member may twist about its longitudinal axis.

These failures occur at loads less than those indicated as being critical by Euler's formula, and at comparatively low values of the slenderness ratio. It is the purpose of this paper to give a theoretical treatment of the subject, together with the results of tests on column sections exhibiting this type of failure. The work is done with duralumin for a material, and on the assumption that the metal is isotropic, with similar elastic properties in all directions.

For theoretical considerations, it is usual to assume that the thin projecting flange or leg is approximated by a flat plate, simply supported at the top and bottom, where the load is applied, and free or under various degrees of fixity at the other two edges. The exact conditions depend on the part of the section which the plate represents.

The theoretical approaches to this problem include an exact mathematical analysis, in which a differential equation for the deflection of the plate from its plane is obtained and then solved for conditions giving the critical stress. An alternative is a method of energy, in which the internal work of elastic deformation is computed. This is then equated to the external work done by the applied force in order to find the critical load.

Previous work done in this direction is mentioned in the historical discussion which follows.

HISTORICAL DISCUSSION

The problem of the buckling of a rectangular plate under edge thrusts in its plane has occupied the attention of a number of investigators since about a decade before the beginning of the twentieth century. The first to take up this subject was G. H. Bryan, who, in the "Proceedings" of the London Mathematical Society (1890, Vol. 22, p.54) treated the simple case of a plate under thrust on both of its edges, which were simply supported. In connection with this article, H. Reissner, in a paper entitled, "Über die Knicksicherheit ebener Bleche," Zentralblatt der Bauverwaltung, (1909) Vol. 29, No. 14, notes that Bryan received some ideas on method (which was that of internal and external work) from Navier. This paper of Bryan's seems to be the first, however, and is mentioned by most of the later writers, who give him the credit for the original solution of the problem.

Following the work by Bryan, the subject evidently dropped from sight for almost twenty years, until S. Timoshenko published several notes in Russian in the yearbooks of the Technical Institutes of St. Petersburg (1906-1907) and of Kiev (1907-1908.) These treatments appeared later in German publications as follows: In an article entitled, "Einige Stabilitätsprobleme der Elastizitätstheorie," in the Zeitschrift für Mathematik und Physik, (1910) No. 58, p. 337, Timoshenko gives several solutions arrived at by the exact mathematical method. His treatment includes the cases where the edge thrust is applied at two of the edges, the third is free, and the fourth is either simply supported, or has various degrees of fixity up to the complete state. The methods in this paper are rigorous, and become complicated in the more advanced cases. Several years later, in the Annales des Ponts et Chaussées, (1913) No. 17, p. 372, the same author has a paper entitled, "Sur la Stabilité des Systèmes Élastiques," in which he attacks the same problems by the method of internal and external work, and arrives at the same conclusions for the cases treated before. In addition, the method is extended to cover more complicated conditions of loading, in which the edge thrust varies from point to point.

In 1909, H. Reissner published the article mentioned earlier in this section, in which he independently arrived at the same results as Timoshenko for the plate which was free on one edge and under different degrees of fixity on the other. Reissner's method was much the same as Timoshenko's earlier one, including as it did the integration of the differential equation for the deflection. Reissner treated four cases, including those noted above and Bryan's case, in which all four sides were simply supported.

Several Years later, the Engineering Record, Vol. 68, No. 26, (Dec. 1913) on page 722 had an article by R. J. Roark, "The Strength of Outstanding Flanges in Beams and Columns." This paper dealt with the subject in a somewhat approximate manner, and the development included a number of assumptions. Since this treatment will not be mentioned again, it will be described a little more completely than the others at this point.

Roark assumed that the outstanding flange could be considered a row of columns if divided vertically, and a row of cantilever beams with perfect fixity at the root end if divided horizontally. He then calculated the load carried as an Euler column, i.e. as if the flange were all by itself. To this was added the load that could be resisted by the combined reaction sideways of the cantilevers, due to a moment at the root of these. By assuming that the elastic curves in both directions were sections of parabolas, and that the flange was held with absolute fixity, Roark was able to integrate over the root to find the total bending moment. This lead to a result in which the critical stress was a function of the thickness and width of the flange, but which included a constant multiplier which did not check very well experimentally. In a later paper in the same publication, Vol. 74, No. 20, (Nov. 1916) Roark revised his assumptions to the effect that the curve of vertical deflection was the sine curve of Euler's theory, and that the horizontal strips assumed the curvature of a cantilever beam under a triangular disposition of the load. These led to a reduction of the constant multiplier. At the end of the article were included results of tests on T and star sections of large size. Most of the measured critical stresses were below the calculated results, and only fair agreement was obtained. Roark noted, however, that when the flange was not perfectly fixed at the edge, the critical stress was reduced.

During the World War, nothing was apparently done on the problem, and it next appeared in 1921, when Timoshenko again published a paper, "Über die Stabilität Versteifter Platten," in Der Eisenbau, Vol. 12, p. 147. In this, he gave a recapitulation of his developments in the 1913 paper and extended the theory further to cover the cases in which the plates had shearing forces along their edges and when there were stiffeners present.

A year later an article by H. M. Westergaard appeared in the "Transactions" of the American Society of Civil Engineers, No. 85, p. 576, entitled, "Buckling of Elastic Structures." This treatment, occupying 100 pages, includes many other problems besides the buckling of plates. There are several sections dealing with the theories discussed by Bryan and Timoshenko.

About this time, the first data on the compression of duralumin columns was obtained. This was included in two theses at the Massachusetts Institute of Technology, one in 1920 and one in 1922. The results of these were discussed in N. A. C. A. Technical Note No. 208, "Tests on Duralumin Columns for Aircraft Construction," by J. G. Lee. This report gives a number of curves, and suggests empirical formulas for use with different lengths of column.

Additional test data was supplied in 1927 by R. A. Miller, whose Air Corps Information Circular No. 598, "The Compressive Strength of Duralumin Channels," advances a semi-empirical theory for the determination of the critical stress in that type of section.

In 1930, the Navy Department published the results of tests by E. M. Krein in Report PTL - 12, "Tests of Aluminum Alloy Columns - Flat Plates and Structural Shapes."

N. A. C. A. Technical Report 382, "Elastic Instability of Members having Sections Common in Aircraft Construction," by G. W. Trayer and H. W. March (1931) seems to be the first instance where the formulas of elastic theory have been applied to experimental results. The work was done on wood, and the theoretical treatment extended Timoshenko's developments to cover non-isotropic materials.

In March, 1932, the N. A. C. A. published Technical Note No. 413, by E. E. Lundquist, on "The Compressive Strength of Duralumin Colimns of Equal Angle Section." This report applies the theoretical formulas to test values for angle sections, and supplies a column chart for the determination of the critical stress.

The theories of Bryan and Timoshenko are mentioned briefly in several text books, among them Love's "Mathematical Theory of Elasticity," Timoshenko's "Applied Elasticity" and "Strength of Materials," and Nadai's "Die Elastichen Platten." The last named work discusses a method whereby the principles of the Calculus of Variations are applied to the expression for the internal work in order to find the critical stress.

PART I

MATHEMATICAL TREATMENT

Since the following work is on the elastic stability of plates, any discussion must be prefaced by a few remarks on stability and the mathematics of plates in general.

In connection with stability, Poincaré[1] has an excellent note on the stability of elastic systems in a concept which he calls the "equilibrium of bifurcation." If we consider the rectangular plate to be subjected to edge loads of magnitude, P, and that its dimensions and physical constants remain the same, the form assumed by the plate is determined by the extension, e, at the edges where the load is applied, and the curvature of the plate, a. Both these quantities are functions of P, and the state of the plate may be represented by a point, with co-ordinates (e, a), which describes a curve as P assumes different values.

When P is less than the critical load, $a = 0$, and the equilibrium state, as defined by $e = f(P)$ is stable. If P exceeds the critical load, $a = 0$ would still give a possible solution and state of equilibrium, but there is also one in which $a \neq 0$, and where a and e are determinate functions of P, so that the equilibrium states for different values of P may be represented by points of a certain curve. This curve branches off from the one given by $a = 0$ at the point of the critical load, and is described by Poincaré as the "point of bifurcation." At this point an exchange of stabilities occurs, and the states represented by the line $a = 0$

1 Acta Mathematica, Tome 7 (1885)
Love, "Elasticity"

now become unstable, and the states represented by the curve $a \neq 0$ become stable.

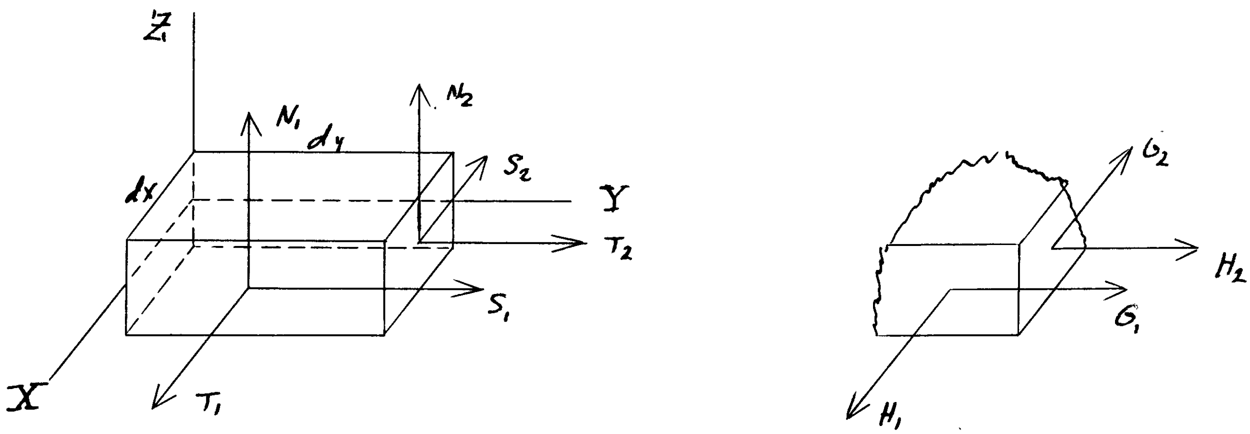
Another theory of elastic stability is due to Southwell.[1] In this, we consider the series of positions in which the plate can be held by a gradually increasing P. If the contracted plate (still in its plane) is given a slight displacement, and the value of P is such that this displacement can be maintained without any change in P, then the equilibrium in this contracted state is critical, and any further increase in P will result in buckling of the plate. Both these concepts fulfill the physical conditions, and can be allied to the theoretical procedure.

In the succeeding work, the following notation will be used, since it has become more or less standard with writers on subjects connected with elasticity.

Symbol	Quantity
E	Modulus of Elasticity
σ , $1/m$	Poisson's Ratio
x, y, z	Co-ordinates on orthogonal axes
$\sigma_x, \sigma_y, \sigma_z$	Normal Stress on any plane
τ	Tangential Stress on any plane
u, v, w	Displacements in x, y, z directions

1 Phil. Trans. Royal Society (1913)
Love, "Elasticity"

We now consider an element of a plate of finite thickness, t. This element has sides of lengths dx and dy. Following the conventions of elastic theory, the expressions for the end forces, and torsional and flexural couples are found from the figure below:



These are grouped as follows:

$$T_1 = \int_{-\frac{t}{2}}^{+\frac{t}{2}} \sigma_X \, dz \qquad S_1 = \int_{-\frac{t}{2}}^{+\frac{t}{2}} \tau_z \, dz \qquad N_1 = \int_{-\frac{t}{2}}^{+\frac{t}{2}} \tau_{Yz} \, dz$$

$$T_2 = \int_{-\frac{t}{2}}^{+\frac{t}{2}} \sigma_Y \, dz \qquad S_2 = -\int_{-\frac{t}{2}}^{+\frac{t}{2}} \tau_z \, dz \qquad N_2 = \int_{-\frac{t}{2}}^{+\frac{t}{2}} \tau_{Xz} \, dz$$

$$H_1 = -\int_{-\frac{t}{2}}^{+\frac{t}{2}} z \tau_z \, dz \qquad G_1 = \int_{-\frac{t}{2}}^{\frac{t}{2}} z \sigma_x \, dz$$

$$H_2 = \int_{-\frac{t}{2}}^{+\frac{t}{2}} z \tau_z \, dz \qquad S_2 = \int_{-\frac{t}{2}}^{+\frac{t}{2}} z \sigma_Y \, dz$$

From these it follows that: $S_1 = -S_2$ and $H_1 = -H_2$

The evaluation of these stress resultants and stress couples in terms of expressions containing the deflection from the x-y plane is accomplished by the following relations between the stresses and the deflection, w, which are fundamental in the theory of elasticity.

$$\sigma_x = -\frac{m^2 E}{m^2-1}\, z\left(\frac{\partial^2 w}{\partial x^2} + \frac{1}{m}\frac{\partial^2 w}{\partial y^2}\right)$$

$$\tau_z = \tau_{xy} = -z\,\frac{mE}{m+1}\,\frac{\partial^2 w}{\partial x \partial y}$$

$$\sigma_y = -\frac{m^2 E}{m^2-1}\, z\left(\frac{\partial^2 w}{\partial x^2}\frac{1}{m} + \frac{\partial^2 w}{\partial y^2}\right)$$

These values are now substituted into the integrals for T, S, H, G, and N, which are then evaluated.

For T:

$$T_1 = -\frac{m^2 E t^2}{2(m^2-1)}\left(\frac{\partial^2 w}{\partial x^2} + \frac{1}{m}\frac{\partial^2 w}{\partial y^2}\right) \quad ; \quad T_2 = -\frac{m^2 E t^2}{2(m^2-1)}\left(\frac{1}{m}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

For S:

$$S_1 = -S_2 = -2\,\frac{mEt}{m+1}\,\frac{\partial^2 w}{\partial x \partial y}$$

For G:

$$G_1 = -\frac{1}{12}\,\frac{m^2 E t^3}{(m^2-1)}\left(\frac{\partial^2 w}{\partial x^2} + \frac{1}{m}\frac{\partial^2 w}{\partial y^2}\right) \quad ; \quad G_2 = -\frac{1}{12}\,\frac{m^2 E t^3}{(m^2-1)}\left(\frac{1}{m}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

For H:

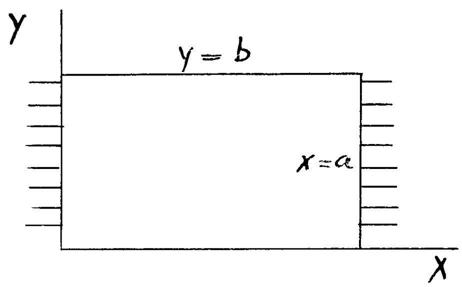
$$H_1 = -H_2 = -\frac{1}{2}\,\frac{mE}{m+1}\,\frac{\partial^2 w}{\partial x \partial y} = -\frac{1}{12}\,\frac{mEt^3}{m+1}\,\frac{\partial^2 w}{\partial x \partial y}$$

The discussion of N is taken up later.

For purposes of abbreviation, the constant multiplier of G and H, is usually referred to as the "cylindrical rigidity" of the plate, and is really the edgewise moment of inertia of a strip of the plate. We have:

$$C = \frac{m^2 E t^3}{12(m^2 - 1)} = \frac{E t^3}{12(1 - \sigma^2)}$$

The ensuing treatment follows in its essentials that given by Timoshenko in his first German paper. The plate is considered to be bounded by the four lines: x = 0, a, and y = 0, b. The end load P, which is the load per unit of length along the edge, is applied to the edges x = 0, a, as indicated. The edge y = b is assumed to be free, and the edge y = 0 to be either simply supported or fixed. In the present case it will be assumed to be simply supported.



Equilibrium between the stress resultants and stress couples in the plate is expressed by the following set of differential equations, the proof of which is not given here.[1]

$$\left.\begin{aligned} &\frac{\partial T_1}{\partial x} - \frac{\partial S_2}{\partial y} - \frac{\partial^2 w}{\partial x^2} N_1 - \frac{\partial^2 w}{\partial x \partial y} N_2 + X' = 0 \\ &\frac{\partial S_1}{\partial x} + \frac{\partial T_2}{\partial y} - \frac{\partial^2 w}{\partial x \partial y} N_1 - \frac{\partial^2 w}{\partial y^2} N_2 + Y' = 0 \\ &\frac{\partial N_1}{\partial x} + \frac{\partial N_2}{\partial y} - \left(-\frac{\partial^2 w}{\partial x^2} T_1 + \frac{\partial^2 w}{\partial x \partial y} S_2\right) + \left(\frac{\partial^2 w}{\partial x \partial y} S_1 + \frac{\partial^2 w}{\partial y^2} T_2\right) + Z' = 0 \end{aligned}\right\} \text{ - (1)}$$

$$\left.\begin{aligned} &\frac{\partial H_1}{\partial x} - \frac{\partial G_2}{\partial y} + N_2 = 0 \quad ; \quad \frac{\partial G_1}{\partial x} + \frac{\partial H_2}{\partial y} - N_1 = 0 \\ &G_1 \frac{\partial^2 w}{\partial x \partial y} - G_2 \frac{\partial^2 w}{\partial x \partial y} + H_1 \frac{\partial^2 w}{\partial x^2} + H_2 \frac{\partial^2 w}{\partial y^2} + S_1 + S_2 = 0 \end{aligned}\right\} \text{ -- (2)}$$

1 Refer to Love, "Elasticity" p. 534

Into equations (2) we substitute the values of G and H from above. This leads to the simplifications indicated below:

$$N_2 = -C \frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)$$

$$N_1 = -C \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)$$

The first two equations of (1) are approximately satisfied if we let $S_1 = S_2 = T_2 = 0$.

Also, from the original conditions of the problem, we have: $T_1 = -P =$ constant, and $X' = Y' = Z' = 0$.

The values obtained are now substituted into the third of equations (1), the last four terms of which vanish immediately, so that the final equation becomes:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{T_1}{C}\frac{\partial^2 w}{\partial x^2} = 0$$

In the solution of this equation for the critical stress, the following boundary conditions must be taken into consideration:

On the simply supported edges; $x = 0, a$, at which the load is applied, we have: $w = 0 \;;\; G_1 = -C\left(\frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial y^2}\right) = 0$

For the simply supported edge $y = 0$:

$$w = 0 \;;\; G_2 = -C\left(\frac{\partial^2 w}{\partial y^2} + \sigma\frac{\partial^2 w}{\partial x^2}\right) = 0$$

On the free edge, $y = b$, the conditions are:

$$G_2 = -C\left(\frac{\partial^2 w}{\partial y^2} + \sigma\frac{\partial^2 w}{\partial x^2}\right) = 0 \quad ; \quad N_2 + \frac{\partial H_2}{\partial x} = 0$$

On substitution of the value, $T_1 = -P$, into the general equation, we have:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{P}{C}\frac{\partial^2 w}{\partial x^2} = 0$$

This will now be solved for the condition in which the edge, $y = 0$, is simply supported, and the above mentioned boundary conditions apply as stated.

The second of the last set of conditons may be combined with the first of equations (2) to yield the following:

Since $H_1 = -H_2$
$$2\frac{\partial H_2}{\partial x} + \frac{\partial G_2}{\partial y} = 0$$

When the values of G_2 and H_2 are substituted, we have:

$$2(1-\sigma)\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial y^2} + \sigma\frac{\partial^2 w}{\partial x^2}\right) = 0$$

This now reduces to the following:

$$(2-\sigma)\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3} = 0$$

The first set of boundary conditions will be satisfied if we let the deflection:

$$w = K \sin\frac{m\pi x}{a} f(y)$$

where f(y) is some function which must be determined from the remaining conditions. (Note: In the Method of Work, a simpler expression for w was assumed directly from general considerations, and worked equally well.)

An equation in f(y) may be obtained by successive differentiations of w and substitution into equation (3).

$$f''''(y) - 2\left(\frac{m\pi}{a}\right)^2 f''(y) + \left[\left(\frac{m\pi}{a}\right)^4 - \frac{P}{C}\left(\frac{m\pi}{a}\right)^2\right] f(y) = 0$$

The solution of this may be written as follows:

$$f(y) = C_1 e^{-\alpha y} + C_2 e^{\alpha y} + C_3 \cos\beta y + C_4 \sin\beta y$$

From the following two equations, we find that $C_1 = - C_2$ and $C_3 = 0$.

$$w = 0\ ; \quad \frac{\partial^2 w}{\partial y^2} + \sigma \frac{\partial^2 w}{\partial x^2} = 0$$

Introducing hyperbolic functions, we have:

$$f(y) = A \sinh\alpha y + B \sin\beta y$$

This result is substituted into the boundary conditions for the edge $y = b$, which are as follows:

$$\frac{\partial^2 w}{\partial y^2} + \sigma \frac{\partial^2 w}{\partial x^2} = 0 \quad ; \quad (2-\sigma)\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3} = 0$$

At this point, mention may be made of the x- term in the complete solution, which is:

$$w = K \sin\frac{m\pi x}{a}\left[A \sinh\alpha y + B \sin\beta y\right]$$

In this, m indicates the number of half waves into which the plate breaks up, and if it has a value greater than 1. each half wave may be considered to be a shorter section of plate, simply supported at the ends.

When the above value of w is substituted into the first of the y = b boundary equations, and the terms are collected, we have:

$$A\left[\alpha^2-\sigma\left(\frac{m\pi}{a}\right)^2\right]\sinh\alpha b-B\left[\beta^2+\sigma\left(\frac{m\pi}{a}\right)^2\right]\sin\beta b=0$$

When the same value is substituted into the second of the equations, another is obtained, as follows:

$$A\alpha\left[\alpha^2-2\left(\frac{m\pi}{a}\right)^2+\sigma\left(\frac{m\pi}{a}\right)^2\right)\cosh\alpha b$$

$$-B\beta\left[\beta^2+2\left(\frac{m\pi}{a}\right)^2-\sigma\left(\frac{m\pi}{a}\right)^2\right]\cos\beta b=0$$

It now remains to satisfy this pair of equations with a solution other A = B = 0, which would mean that the plate was in equilibrium with no deflection from its plane. This is possible if the determinant of the coefficients of A and B shall vanish, so that we have:

$$0=\begin{vmatrix}\left[\alpha^2-\sigma\left(\frac{m\pi}{a}\right)^2\right)\sinh\alpha b, & -\left[\beta^2+\sigma\left(\frac{m\pi}{a}\right)^2\right\}\sin\beta b] \\ \alpha\left[\alpha^2-2\left(\frac{m\pi}{a}\right)^2+\sigma\left(\frac{m\pi}{a}\right)^2\right]\cosh\alpha b, & -\beta\left[\beta^2+2\left(\frac{m\pi}{a}\right)^2-\sigma\left(\frac{m\pi}{a}\right)^2\right]\cos\beta b\end{vmatrix}$$

The solution of this leads to an equation given on the next page.

$$\beta\left[\alpha^2-\sigma\left(\frac{m\pi}{a}\right)^2\right]\tanh\alpha b=\alpha\left[\beta^2+\sigma\left(\frac{m\pi}{a}\right)^2\right]\tan\beta b$$

This equation is to be solved for the critical stress, after the following substitutions of U and V are made, where they are defined as follows:

$$\alpha b=\sqrt{\sqrt{UV}+V}\qquad U=\frac{Pb^2}{C}$$

$$\beta b=\sqrt{\sqrt{UV}-V}\qquad V=\left(\frac{\pi b}{a}\right)^2$$

This leads to the final equation:

$$\sqrt{\sqrt{UV}-V}\left[(1-\sigma)V+\sqrt{UV}\right]^2\tanh\left(\sqrt{\sqrt{UV}+V}\right)$$

$$=\sqrt{\sqrt{UV}+V}\left[\sqrt{UV}-(1-\sigma)V\right]^2\tan\sqrt{\sqrt{UV}-V}$$

The procedure now is to assume a number of values of V, i.e. for the ratio b/a, substitute them into the above equation, and solve for U. From U, the critical load may be determined.

As is immediately evident, this method, besides being somewhat complicated in itself, leads to an equation which is extremely laborious to solve for a large number of cases. If any change is made in the boundary conditions, the calculations become still more involved. Consequently, some other method of approach is to be desired.

-Method of Elastic Work-

The hypothesis underlying the theory of elastic work of deformation is that no stress acting anywhere in the body under consideration is of an impulsive type, but that the stresses increase from zero to a maximum, so that the average may be taken for the range of values.

Thus, for direct stresses, we have as the work done in the x-direction; $\frac{1}{2}\sigma_x \epsilon_x$ and for tangential stresses, the work of distortion from any single shear stress is: $\frac{1}{2}\gamma_x \tau_x$

Since these expressions are for a unit element only, the total value of the internal work is obtained by summing these up over the whole body, as follows:

$$W_i = \frac{1}{2}\iiint\left[\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z + \tau_x \gamma_x + \tau_y \gamma_y + \tau_z \gamma_z\right\} dx\,dy\,dz$$

If we substitute the following expressions, which are fundamental in elastic theory, we may obtain an expression for W_i in terms of the stresses alone.

$$\epsilon_x = \frac{1}{2G}\left(\sigma_x + \frac{\theta}{m+1}\right) \qquad \text{where} \qquad \theta = \sigma_x + \sigma_y + \sigma_z$$

$$\gamma_x = \frac{\tau_x}{G} \qquad \text{and} \qquad G = \frac{mE}{2(m+1)}$$

The above equation reduces to another one, as shown.

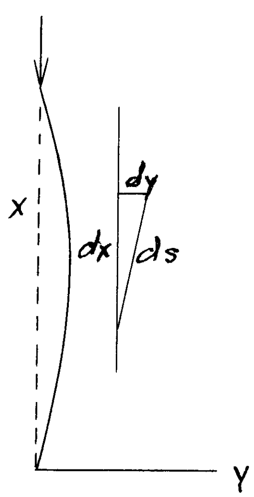
$$W_i = \frac{1}{2G}\iiint \frac{1}{2}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\theta^2}{2(m+1)} + (\tau_x^2 + \tau_y^2 + \tau_z^2)\Big\} dx\,dy\,dz$$

In the case of the plate, we have only σ_x, σ_y, τ_z, which have the same values they had earlier. This simplifies the equation for the internal work, and leads to the final form.

$$W_i = \frac{1}{2E}\iint\left(\sigma_x^2+\sigma_y^2+\frac{2}{m}\sigma_x\sigma_y\right)dx\,dy + \frac{1}{2G}\iint \tau_z^2\,dx\,dy$$

$$W_i = \frac{C}{2}\int_0^a\!\!\int_0^b \left[\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\sigma)\left\{\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right\}\right]dx\,dy$$

In the problem of the evaluation of the external work, we have a force, P, which is constant during the interval in which it acts, so that the factor ½ does not enter in this case in connection with P. If u is the displacement, the external work on a lengthwise strip of the plate is given by the expression below.



$$W_e = P\,u\,dx$$

From the figure on the left, u is calculated as follows:

$$u = \frac{ds - dx}{dx} = \sqrt{1-\left(\frac{\partial y}{\partial x}\right)^2} - 1$$

Expanding and dropping terms of higher order, we have:

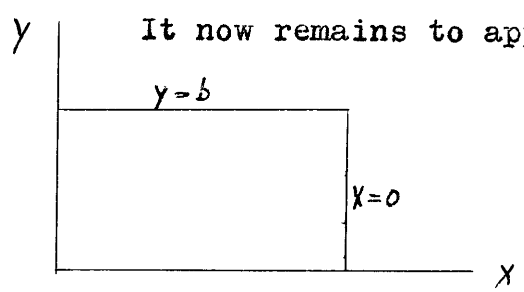
$$u = \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2$$

The total value of the external work on the plate becomes the double integral below.

$$W_e = \frac{1}{2} P \int_0^a \int_0^b \left(\frac{\partial w}{\partial x} \right)^2 dy\, dx$$

It may be remarked that these expressions for the internal and external work are in reality second order quantities relative to any sidewise displacement, and that the work done by stress forces and external forces is not of first order.

When W_i and W_e are equated, we have mutual equilibrium. If the load is small, an excess of energy due to bending in the plate makes it return to its former position. When W_e is greater, the deflection tends to increase. The equality of these two expressions for the work is the criterion of instability, and the load which corresponds to this condition is the critical load.



It now remains to apply this theory to the plate of sides, x = 0, a, and y = 0, b which was discussed earlier. The problem is that of assuming a suitable expression for the deflection, w. Since three of the sides are simply supported, and the fourth is free, the following value fulfills the boundary conditions satisfactorily.

$$w = K y \sin \frac{\pi x}{a}$$

This presupposes that we have a rotation of each elemental strip about the axis of x.

The expression for w is now substituted into the complete expression for the internal work.

Upon successive differentiations of w, several terms of the integral drop out when the values are substituted, and we have:

$$W_i = \frac{C}{2}\int_0^a\int_0^b\left[\left(-\frac{\pi^2}{a^2}Ky\sin\frac{\pi x}{a}\right)^2 - 2(1-\sigma)\left\{-\left(\frac{\pi}{a}K\cos\frac{\pi x}{a}\right)^2\right\}\right]dx\,dy$$

More terms vanish in the integration between the limits 0 and a, and 0 and b, and the final result is as follows:

$$W_i = K^2\pi^4\frac{C}{2}\left[\frac{b^3}{6a^3} + \frac{(1-\sigma)}{\pi^2}\frac{b}{a}\right]$$

For the value of the external work, the evaluation of the integral yields the final result.

$$W_e = \frac{P\pi^2 b^3 K^2}{12a}$$

The internal and external work are now equated to find the critical solution.

$$P = \left[\frac{\pi^2}{a^2} + \frac{6(1-\sigma)}{b^2}\right]C$$

This result may be correlated to the solution given by the exact method if we introduce the expression, U.

$$P = \frac{C\,U}{b^2} \quad ; \quad U = \frac{\pi^2 b^2}{a^2} \quad 6(1-\sigma)$$

If values of the physical characteristics and dimensions of the plate under consideration are substituted in and a specific U is found, its amount is exactly that found from substitution of values of V into the equation provided by the exact method.

The method of internal and external work will now be extended to cover the case in which four sides of the plate are simply supported, and the forces are applied as before, on the two edges: x = 0, and x = a.

In this case the deflection, w, may be represented by a double sine series, as follows:

$$w = \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

In this, m indicates the number of half waves into which the plate divides as it deflects. If this expression is substituted into the integral for the internal work, which is then solved as before, we have, after some reduction:

$$W_i = \frac{C}{2} \frac{ab}{4} \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} A_{mn}^2 \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right)^2$$

The same value is substituted into the equation for the external work, which becomes:

$$W_e = \frac{P}{2} \frac{ab}{4} \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} A_{mn}^2 \frac{m^2\pi^2}{a^2}$$

W_e is now set equal to W_i as before, and an expression for the critical load is obtained.

$$P = \frac{C \sum_1 \sum_1 A_{mn}^2 \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right)^2}{\sum_1 \sum_1 A_{mn}^2 \frac{m^2\pi^2}{a^2}}$$

In applying this, we set n equal to 1, and chose m so that the critical load will be a minimum. The formula then simplifies to the following:

$$P = \frac{C\pi^2 \left(\frac{m^2}{a^2} + \frac{1}{b^2} \right)^2}{\frac{m^2}{a^2}}$$

If the length of the plate is small, so that it divides into only one half wave, the value of the load becomes:

$$P = C\pi^2 \left[\frac{1}{a} + \frac{a}{b^2} \right]^2$$

The question now arises as to when the plate will break up into a single half wave, and when into two or more. Since, at the point of transition from m waves into m + 1 waves, the critical load is the same, we may obtain an equation as follows:

$$\frac{\left(\frac{m^2\pi^2}{a^2} + \frac{\pi^2}{b^2} \right)^2}{\frac{m^2\pi^2}{a^2}} = \frac{\left(\frac{(m+1)^2\pi^2}{a^2} + \frac{\pi^2}{b^2} \right)^2}{\frac{(m+1)^2\pi^2}{a^2}}$$

This reduces to the relation between a and b: $a = b\sqrt{m(m+1)}$

So, as m increases, we have the following relations a and b.

m:	1	2	3	4	5
a/b:	1.4	2.45	3.46	4.47	5.48

Investigation shows that the constant for any particular case:

$$\frac{b^2\left[\frac{m^2}{a^2}+\frac{1}{b^2}\right]^2}{\frac{m^2}{a^2}}$$

Has values slightly above 4 at low m, but as m increases above 2 and 3, the value 4 is a very close approximation. The expression for the critical load for long plates then becomes:

$$P = \frac{4\pi^2 C}{b^2}$$

If a/b is below 2.5, the longer equation should be used.

- Discussion of Theoretical Results -

In the formulas which have been developed, it is important to note that P is the critical running load, so that the critical stress is given by:

$$p = \frac{P}{t} \quad \text{in whatever units are used.}$$

Furthermore, in both the expressions it is evident that as the length of the plate, a, increases, the critical load tends towards a constant value.

For the plate with one edge free, the term, $\frac{\pi^2}{a^2}$ becomes progressively smaller, while the rest of the terms remain constant, so that the curve of P against a drops sharply at first, and then flattens out horizontally.

For the plate with both edges simply supported, the curve of P against a behaves in a similar manner, as has already been noted.

PART II

APPLICATION OF THEORETICAL FORMULAS TO TEST RESULTS

The results in the previous section have been obtained for plates which had various mathematical boundary conditions, and to which the load was considered to be uniformly applied. In relating the formulas to duralumin columns, it is evident at once that there is chance for divergence between the actual test conditions and those of the theory, particularly in the amount of fixity of the supported leg of the column, to which the plate formulas are applied.

In respect to the various degrees of edge fixity, it may be mentioned that complete support at the edges almost never occurs, at least in the type of columns used in duralumin construction. This would require that the outstanding flange should project from a practically solid mass of metal at its base, which is a condition not met with in practice.

The simplest column section to which theory may be applied is an angle whose legs have the same dimensions, and may be considered to be plates which are simply supported on their inner edges. Channels, Z-, and other sections become more complicated.

The test results used here were obtained at various times by different people, and the work from this point on will divided according to the type of section and the source of the test results.

A remark is in order at this point on the range of values of the slenderness ratio through which the plate formulas may be used. At high values of L/r, the column tends to fail as a whole, by bending, rather than in any of its parts, and the critical stress is given by Euler's formula:

$$\text{Stress} = \frac{c\,\pi^2 E}{(L/r)^2}$$

in which r is the least radius of gyration of the section. The plate formulas have no significance when applied to columns in the Euler range, since at that point, the mechanism of failure becomes entirely different.

In the tabulations which follow, the following symbols are used:

L = length of column in inches

S = width of an outstanding leg in inches

B = width of back of a channel in inches

All loads are in pounds, and all stresses in pounds per square inch.

ANGLES

Most of the tests run on columns of angle section have been on those in which both legs had the same dimensions, so that for the theoretical work, they could be considered to be two identical plates, simply supported on one edge and free on the other. The expression used was:

$$P = C\left[\frac{\pi^2}{a^2} + \frac{6(1-\sigma)}{b^2}\right] \quad \text{where} \quad C = \frac{E\,t^3}{12(1-\sigma^2)}$$

The following test results were obtained from a thesis written by Messrs. Harsch and Whitehead at M. I. T. in 1920. In this, the columns were tested as pin ended columns by the use of spherical bearings, fully described in their paper. Several different types of sections were used, including channels, modified angle sections, and angles with equal legs, the last of which are of interest here. A number of different sizes of each section were tried, and at different values of L/r. The comparative results of experiment and theory are given on the first of the tables and curves which follow, together with the dimensions of the sections.

In this same thesis, some data is given for British tests on similar sections and under the same conditions. The application of theory to these is also given here.

Krein, in the Navy Report mentioned in the historical discussion has some results of a few tests on angles, which are worked out at the end of this section.

Harsch & Whitehead Mark S-1

E = 10 200 000 σ = 3/10

t = .0417 S = .788 S/t = 18.87 A = .061

L/r	L	C	Calc. Load	Calc. Stress	Test Stress
19.19	3.12	67.7	830	13600	17800
40.7	6.38	67.7	748	12270	13000
60.9	9.55	67.7	734	12000	12130
80.6	12.65	67.7	728	11940	11000
101.9	16.0	67.7	726	11900	8460

Harsch & Whitehead Mark S-2

E = 10 200 000 σ = 3/10

t = .0505 S = .788 S/t = 15.6 A = .076

L/r	L	C	Calc. Load	Calc. Stress	Test Stress
19.9	3.12	120.3	1476	19430	23160
40.5	6.35	120.3	1328	17500	17760
60.5	9.5	120.3	1302	17100	15260
82.0	12.82	120.3	1294	16800	10500
101	15.98	120.3	1290	16750	7700

Harsch & Whitehead Mark S-3

E = 10 200 000 σ = 3/10

t = .0579 S = .788 S/t = 13.6 A = .0845

L/r	L	C	Calc. Load	Calc. Stress	Test Stress
21.9	3.42	181.2	2170	25700	25600
40.3	6.27	181.2	2000	23700	22500
61.4	9.59	181.2	1960	23200	18500
82.0	12.82	181.2	1950	23100	13100
101	15.93	181.2	1944	23000	8930

British Mark N

E = 10 700 000 σ = 3/10

t = .05 S = 1.0 S/t = 20 A = .0975

L/r	L	C	Calc. Load	Calc. Stress	Test Stress on curves
10	1.97	122.4	1652	16950	
20	2.84	122.4	1328	13630	20000
30	5.91	122.4	1100	11300	15000
40	7.88	122.4	1066	10940	12500
60	11.83	122.4	1044	10700	11500
80	15.75	122.4	1038	10650	11400
100	19.7	122.4	1038	10650	10500

British Mark L

E = 10 700 000 σ = 3/10

t = .04 S = .7 S/t = 17.5 A = .0544

L/r	L	C	Calc. Load	Calc. Stress	Test Stress on curves
10	1.37	62.7	1216	22300	
20	2.76	62.7	956	17540	
30	4.11	62.7	804	14760	
40	5.48	62.7	796	14620	
60	8.22	62.7	764	14010	
80	10.95	62.7	760	13950	
100	13.7	62.7	756	13900	

Krein Table III

E = 10 000 000 σ = 3/10

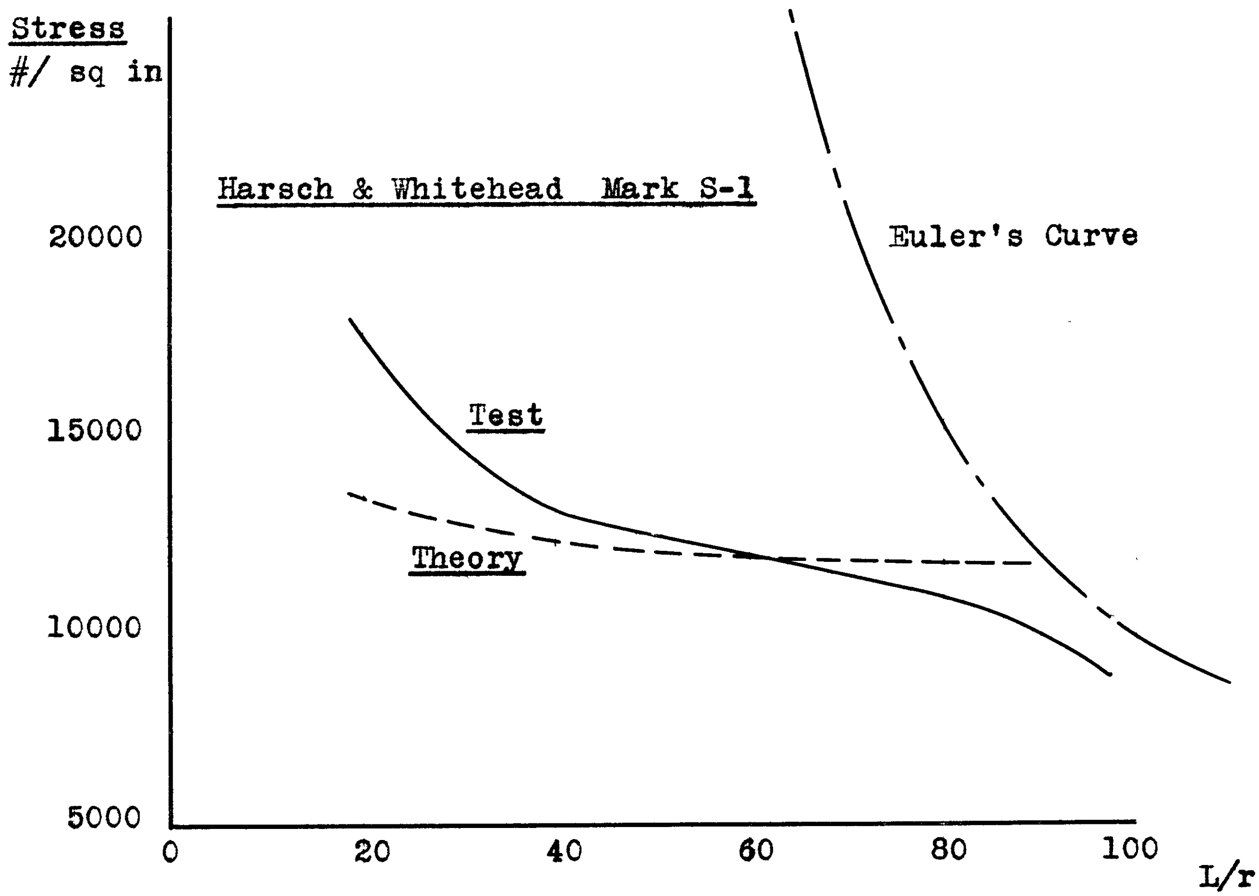
t = .029 S = .75 S/t = 25.8 A = .0426

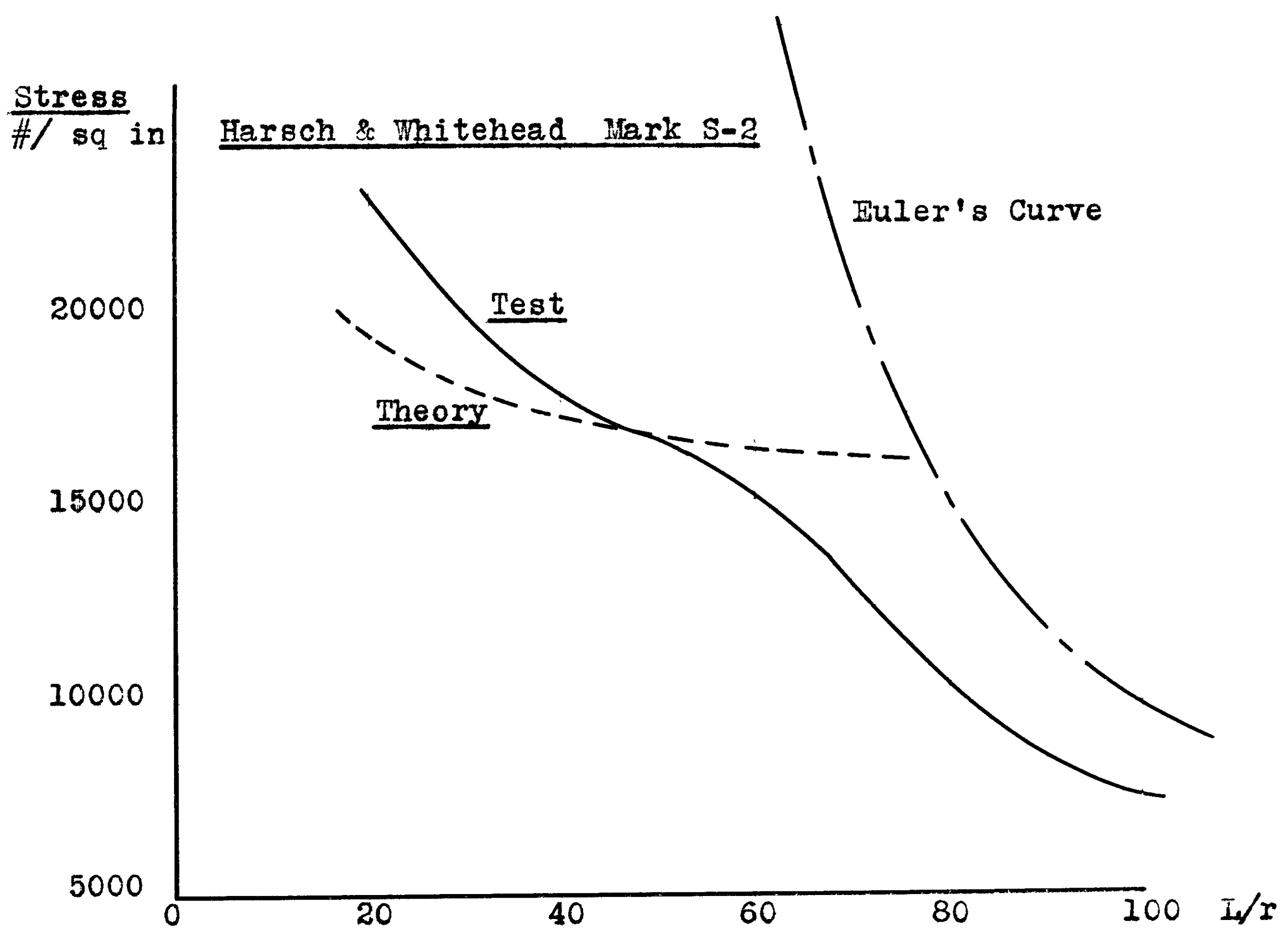
L/r	L	C	Calc. Load	Calc. Stress	Test Stress
36.1	6	22.3	259	6070	8120
72	12	22.3	252	5820	6160

E = 10 000 000 σ = 3/10

t = .081 S = .75 S/t = 9.26 A = .1149

L/r	L	C	Calc. Load	Calc. Stress	Test Stress
44.5	6	486.0	5640	49000	24800
88.8	12	486.0	5480	47700	10600


Stress
#/ sq in
Harsch & Whitehead Mark S-1
Euler's Curve
Test
Theory
20000
15000
10000
5000
0
20
40
60
80
100
L/r


Stress
#/ sq in
Harsch & Whitehead Mark S-2
Euler's Curve
Test
Theory
20000
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10000
5000
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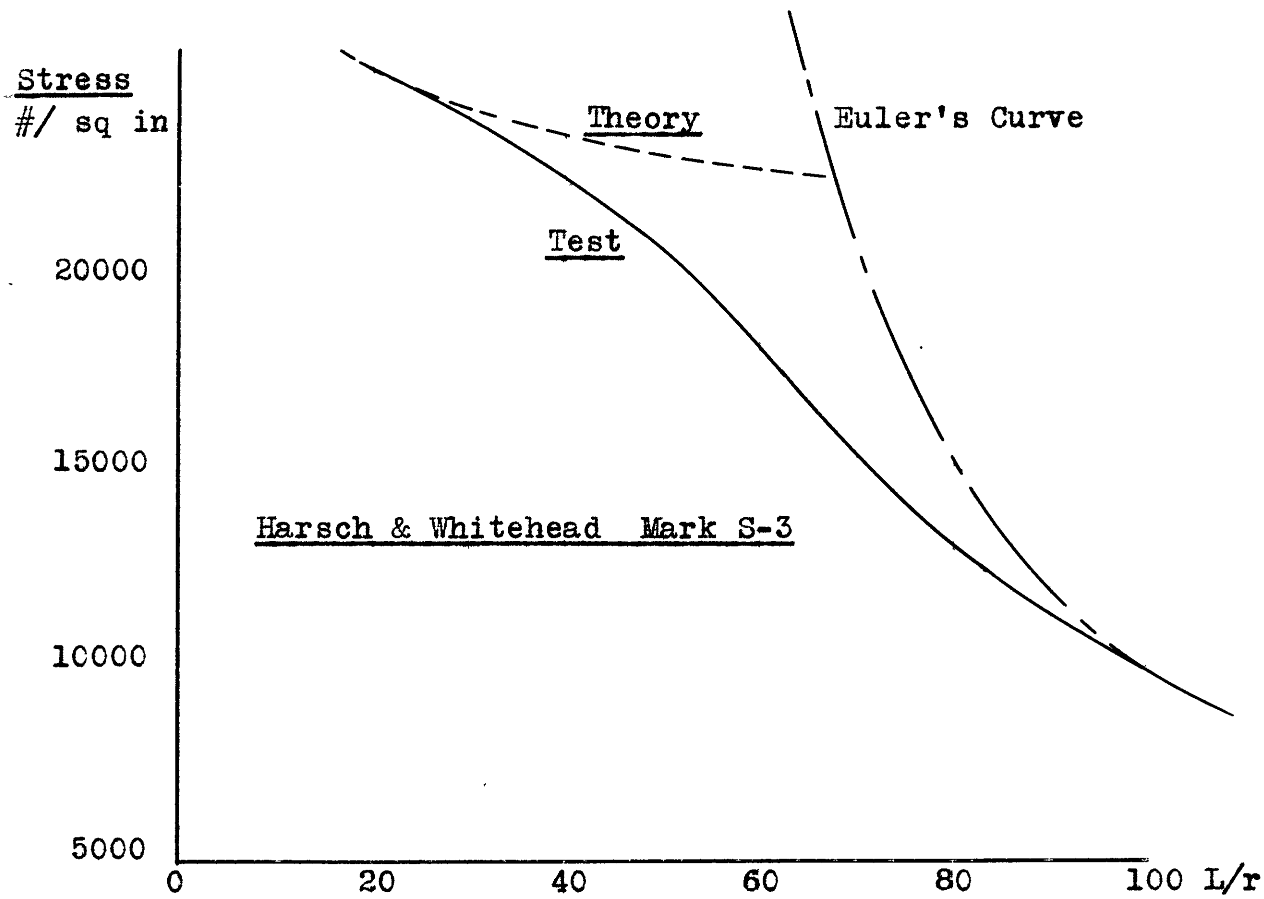
20
40
60
80
100
L/r



Stress
#/ sq in
20000
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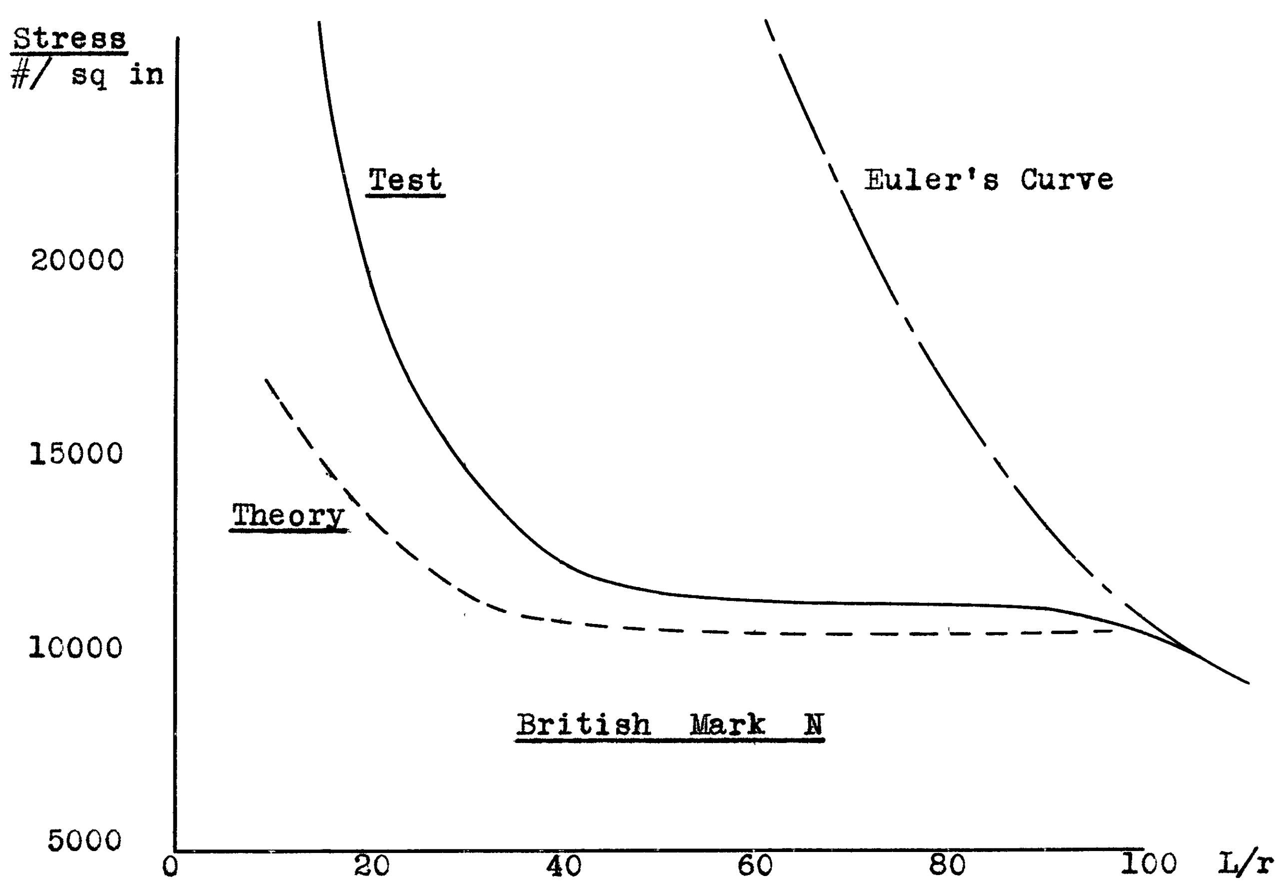
5000
0
20
40
60
80
100 L/r
Theory
Euler's Curve
Test
Harsch & Whitehead Mark S-3



Stress
#/ sq in
20000
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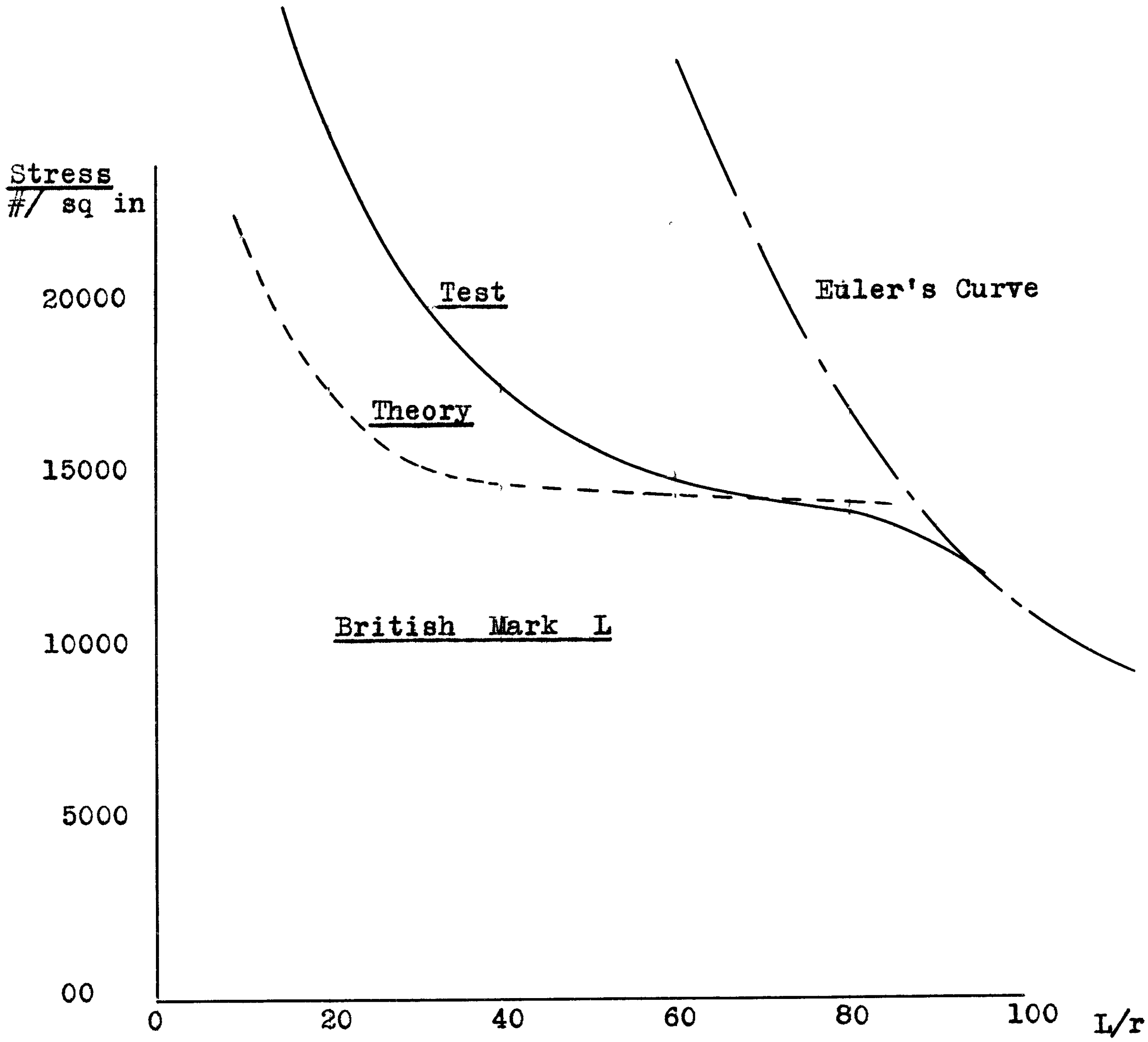
L/r
Test
Euler's Curve
Theory
British Mark N


Stress
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20
40
60
80
100
L/r
Test
Theory
Euler's Curve
British Mark L

DISCUSSION OF RESULTS AND CONCLUSIONS

In connection with these results, it may be mentioned that Lundquist (N. A. C. A. T. N. 413) applied the theoretical formula for angles to some of the data included here, and in addition to some other British Tests. He arrived at the same conclusions indicated by the curves given here - that the theory is over conservative at low values of L/r, though it gives a good approximation in the range of L/r from 40 to 80.

The conditions on which the mathematics of the theory is built up seem to be most closely fulfilled when S/t is from 20 to 25. At lower values of this ratio, the theory tends to overestimate the critical stress.

When S/t is from 20 to 25, and L/r is from 40 to 80, it may be concluded that the theory will give good results. At low L/r it is likely to be conservative, and at medium values of L/r, with S/t very low, the results are too high.

CHANNELS

The most valuable data for this section was that taken from A. C. I. C. No. 598, "The Compressive Strength of Duralumin Channels," by R. A. Miller. In general, these channels are rather larger than most of the extruded types, and were fabricated in a brake or by hammering with a wooden mallet over a hard wood block while the material was in the annealed condition. It was afterward heat-treated.

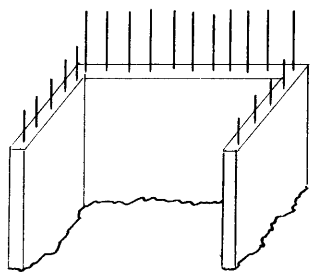
In the tests, the ends of the specimens were supported in cradles, which were in turn mounted on knife edges at the same distence apart as the ends of the channels. In this way, they were tested as pin-ended columns. Any eccentricity of the load, which was apparent as a deflection, was removed by adjustment of micrometer screws on the cradles, which could be shifted slightly so that the compressive load was applied in the neutral plane.

Additional data was supplied by Messrs. Becker and Noveck in a thesis written at M. I. T. in 1922. A series of tests were run on three general types of channels, the first of which, the Mark-A, are of most interest here. The end loads were applied through spherical bearings, which are described in their paper, and the effect was that of a pin-ended column.

The Navy Report by Krein, mentioned earlier, also contains a small amount of data on the compressive strength of channel sections. The results of all these experiments are listed in tables which follow.

In applying the theoretical formulas to channel sections, we approximate the two legs by plates, simply supported on the inner edge, and free on the outer edge. The back of the channel is represented by a plate which is simply supported on both of its edges.

It is evident that in any given case, the critical load and critical stress of the legs will not be equal to the critical load and stress of the back, a condition which is represented by



the figure on the left. The question now arises as to what critical stress is to be taken as being that of the column.

Investigation shows that a satisfactory method is to calculate the load which is critical for the legs, and that which is critical for the back. The sum of these loads is then the critical load for the column, and the stress is at once obtained when this number is divided by the area of the whole section.

The above procedure amounts to assuming that where a uniform load is applied to the ends, it distributes itself according to the strength of the part of the section on which it acts. Usually the legs are found to be the weaker part, so that as the load increases, it is taken equally by sides and back up to the point where the critical value for the sides is reached. From that point on, the extra load is carried by the back until the critical point for the back is attained and slightly exceeded, whereupon the column fails.

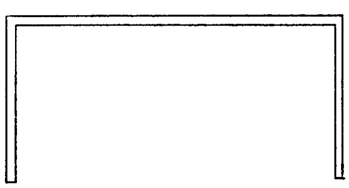
For a channel with equal legs, the expression for the critical stress then becomes:

$$p_{critical} = \frac{2\ L_s + L_b}{A}$$

where L_s and L_b are the critical loads on sides and back respectively.

This method will now be applied to an example from Miller's data, in which the agreement between theory and the test value is fairly close.

Suppose that we have a duralumin channel which is 4.08" in length, having sides of .84" and .85" respectively, a back of 1.75", and a thickness of .053". For this material, $\sigma = 3/10$ and E = 9 730 000.



In applying the formulas, we first find the value of C, which is constant for both legs and the back.

$$C = \frac{E\ \ t^3}{12(1 - \sigma^2)} = \frac{9\ 730\ 000(.000149)}{12(1 - .09)} = 132.7$$

For the running load on the two legs, which is slightly different in each case, we have the expression:

$$P = C\left[\frac{\pi^2}{a^2} + \frac{6(1 - \sigma)}{b^2}\right]$$

$$P_s = 132.7\left[\frac{9.87}{16.65} + \frac{6(0.7)}{.706}\right] = 869; \quad L_s = 869(.84) = 730$$

$$P_{s'} = 132.7\left[\frac{9.87}{16.65} + \frac{6(0.7)}{.723}\right] = 850; \quad L_{s'} = 850(.85) = 722$$

In determining the critical running load for the back of the channel, we have C = 132.7, and since the ratio S/B = a/b = 2.33, the complete constant multiplier developed in the theory is calculated and found to be 4.09 rather than the value of 4. The expression for the load is then:

$$P_b = \frac{132.7(987)4.09}{3.063} = 1755; \; L_b = 1755(1.75)$$

$$L_b = 3075$$

The total critical load then becomes:

$$\begin{array}{r} 730 \\ 722 \\ \underline{3075} \\ 4527\# \end{array} = \text{Load}$$

This checks fairly well with the experimental value of 5000#. If both are divided by the total section area of .1767 sq. ins., we have a calculated critical stress of 25600 #/ sq. in. against a tests stress of 28300 #/ sq. in.

It is to be noted that in channels of ordinary proportions, such as this one, the allowable running load for the legs is much less than the allowable running load for the back.

In the tables which follow, the above procedure is applied to a number of channels of different types and slenderness ratios, the data for which is taken from the sources mentioned.

Miller - Table I

E = 9 730 000 σ = 3/10

L/r	L	S	B	t	S/t	B/t	S/B	C
14.90	5.9	1.25	2.53	.05	25	50.6	.494	111.4
24.97	9.9	1.25	2.50	.051	24.5	49.0	.500	118.1
35.18	13.95	1.25	2.50	.05	25	50	.500	111.4
45.0	17.84	1.25	2.50	.051	24.5	49.0	.500	118.1
55.0	21.8	1.25	2.50	.051	24.5	49.0	.500	118.1
65.1	25.78	1.25	2.50	.051	24.5	49.0	.500	118.1
84.85	33.65	1.25	2.50	.05	25	50	.500	111.4
95.0	37.69	1.25	2.50	.05	25	50	.500	111.4

L/r	Calc. Load	Av. Test Load	Calc. Stress	Test Stress
14.90	2615	3425	10770	14110
24.97	2691	3200	10930	13010
35.18	2522	4005	10460	16605
45.0	2669	3040	10850	12360
55.0	2665	2970	10830	12080
65.1	2661	2715	10820	11040
84.85	2512	2100	10390	8705
95.0	2512	2280	10390	9450

Miller - Table II

E = 9 730 000 σ = 3/10

L/r	L	S	B	t	S/t	B/t	S/B	C
15.47	4.08	.845	1.75	.053	16.0	33.0	.483	132.7
13.55	4.08	.95	1.75	.053	17.9	33.0	.543	132.7
14.19	4.15	.925	1.75	.051	18.1	34.3	.528	118.1
13.62	4.15	.96	1.75	.052	18.45	33.7	.548	125.5
14.25	4.17	.925	1.74	.051	18.1	34.3	.531	118.1
24.46	6.9	.895	1.72	.052	17.2	33.1	.520	125.5
25.45	6.9	.865	1.72	.052	16.5	33.1	.502	125.5
31.0	9.6	.975	1.75	.054	18.05	32.4	.556	140.3
32.45	9.6	.935	1.75	.053	17.44	33.0	.534	132.7
40.6	12.37	.96	1.77	.052	18.5	34.0	.542	125.5
40.4	12.37	.965	1.78	.052	18.55	34.2	.542	125.5
49.4	15.13	.965	1.74	.052	18.55	33.5	.554	125.5
47.2	15.15	1.00	1.72	.052	19.2	33.1	.581	125.5
59.65	17.95	.95	1.76	.051	18.6	34.5	.540	118.1
60.7	17.95	.935	1.76	.052	18.0	33.8	.531	125.5
67.8	20.65	.96	1.75	.052	18.5	33.6	.548	125.5
70.2	20.68	.925	1.75	.051	18.1	34.3	.527	118.1
80.2	23.45	.925	1.75	.052	17.8	33.6	.528	125.5
80.2	23.45	.925	1.74	.051	18.1	34.1	.531	118.1

Miller - Table II (continued)

L/r	Calc. Load	Av. Test Load	Area	Calc. Stress	Test Stress
15.47	4527	5000	.1767	25600	28300
13.55	4400	4570	.1878	23400	24330
14.19	3929	4130	.1784	22100	23150
13.62	4134	3380	.1854	22200	18230
14.25	3955	3520	.1778	22100	19800
24.46	4105	4335	.1771	23100	24480
25.45	4146	3630	.1740	23700	20860
31.0	4502	4170	.1928	23300	21630
32.45	4190	4330	.1862	22500	23250
40.6	3912	4250	.1864	20950	22800
40.4	3890	4080	.1875	20700	21760
49.4	3958	3140	.1854	21300	16940
47.2	3950	4140	.1885	20900	21960
59.65	3700	3350	.1814	20400	18470
60.7	3958	3210	.1833	21600	17510
67.8	3926	3220	.1854	21000	17370
70.2	3745	2940	.1784	21000	16480
80.2	3974	2620	.1818	21600	14410
80.2	3754	2520	.1778	21100	14170

Becker & Noveck - Mark A-1

E = 10 700 000 σ = 3/10

L/r	L	S	B	t	S/t	B/t	S/B	C
20.4	4.73	.745	.965	.035	21.3	27.6	.772	42
40.4	9.38	.745	.965	.035	21.3	27.6	.772	42
60.6	14.08	.745	.965	.035	21.3	27.6	.772	42
80.4	18.62	.745	.965	.035	21.3	27.6	.772	42
100.4	23.3	.745	.965	.035	21.3	27.6	.772	42

L/r	Calc. Load	Av. Test Load	Area	Calc. Stress	Test Stress
20.4	2222	2485	.0816	28500	31850
40.4	2200	2566	.0816	28200	32900
60.6	2196	1783	.0816	28100	22850
80.4	2194	1130	.0816	28000	14500
100.4	2194	766	.0816	28000	9830

Becker & Noveck - Mark A-2

L/r	L	S	B	t	S/t	B/t	S/B	C
20.4	4.70	.745	.965	.05	14.9	19.3	.772	122.4
40.4	9.14	.745	.965	.05	14.9	19.3	.772	122.4
60.3	13.8	.745	.965	.05	14.9	19.3	.772	122.4
80.5	18.5	.745	.965	.05	14.9	19.3	.772	122.4
100	23.0	.745	.965	.05	14.9	19.3	.772	122.4

Becker & Noveck - Mark A-2 (cont.)

L/r	Calc. Load	Av. Test Load	Area	Calc. Stress	Test Stress
20.4	6462	3833	.1167	56400	39340
40.4	6400	3216	.1167	54900	33333
60.3	6390	2166	.1167	54800	23780
80.5	6380	1450	.1167	54700	16500
100	6380	1080	.1167	54700	11873

Calculations for the Becker& Noveck Mark A-3 Sections showed still greater divergence, the ratio of Calculated/Test being about 1.78 at an L/r of 20.1

Krein Table III

E = 10 000 000 = 3/10

L/r	L	S	B	t	S/t	B/t	S/b	C
40.3	7.25	.562	.75	.029	19.37	25.8	.75	22.3
27.7	7.25	.562	.75	.081	6.94	9.25	.75	486

L/r	Calc. Load	Av. Test Load	Area	Calc. Stress	Test Stress
40.3	1512	1050	.0526	28700	19900
27.7	32800	4160	.144	246000	30700

Miller - Table I'

E = 9 730 000 σ = 3/10

L/r	L	S	B	t	S/t	B/t	S/B	C
35	14.28	1.25	1.25	.052	24	24	1.0	125.5
35	14.27	1.25	1.50	.051	24.5	29.4	.833	118.1
35.06	14.13	1.25	2.00	.052	24	38.5	.625	125.5
35.18	13.95	1.25	2.50	.050	25	50	.50	111.4
35	13.6	1.25	3.00	.051	24.5	58.8	.417	118.1
35.06	13.3	1.25	3.50	.051	24.5	68.7	.357	118.1

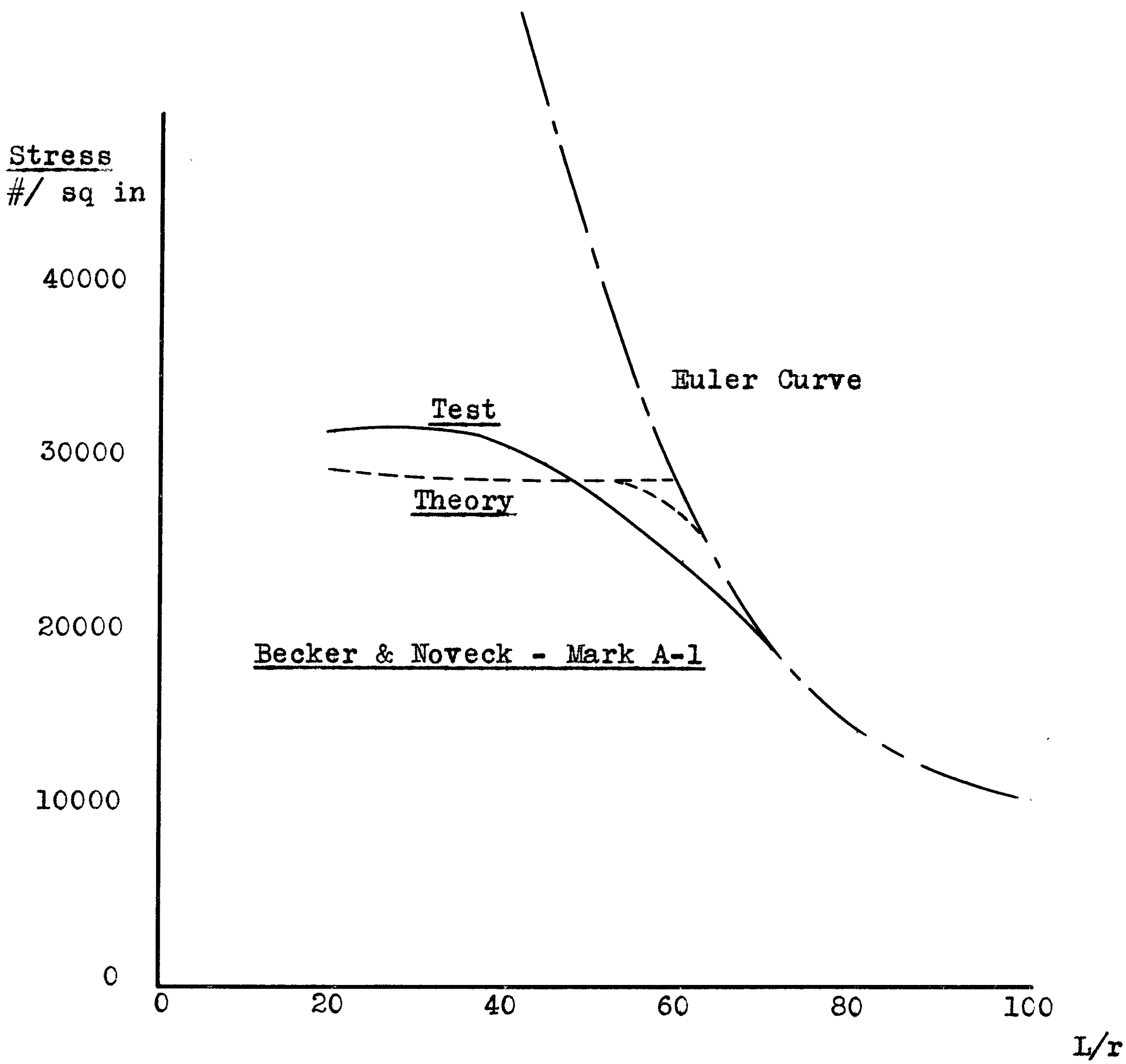
L/r	Calc. Load	Av. Load	Area	Calc. Stress	Test Stress
35	4810	2955	.1856	26400	15920
35	3918	2695	.1949	20100	13830
35.06	3340	3075	.2246	14870	13690
35.18	2524	4005	.2412	10470	16605
35	2362	3675	.2714	8690	13540
35.06	2138	3550	.2969	7200	11955

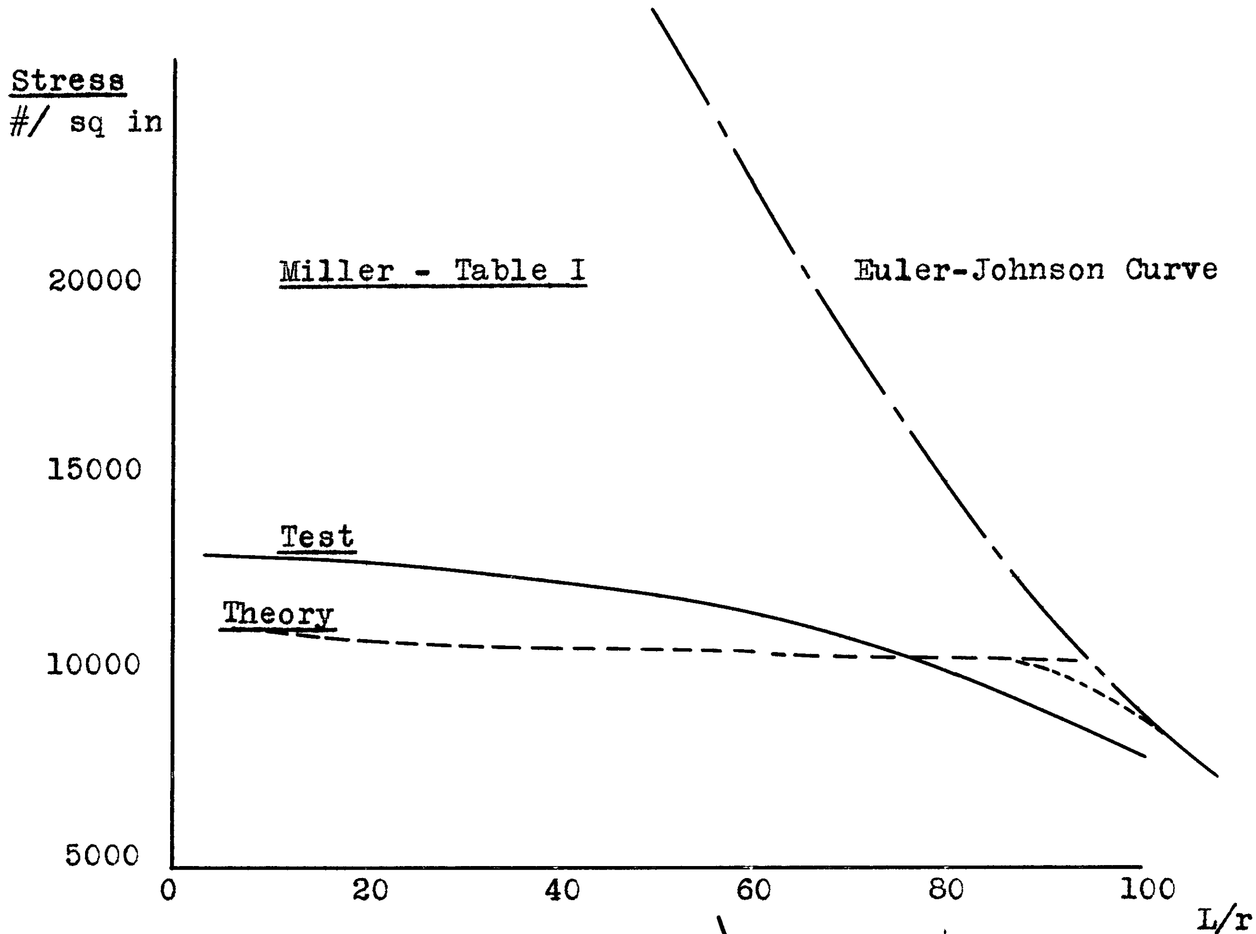
Miller - Table I''

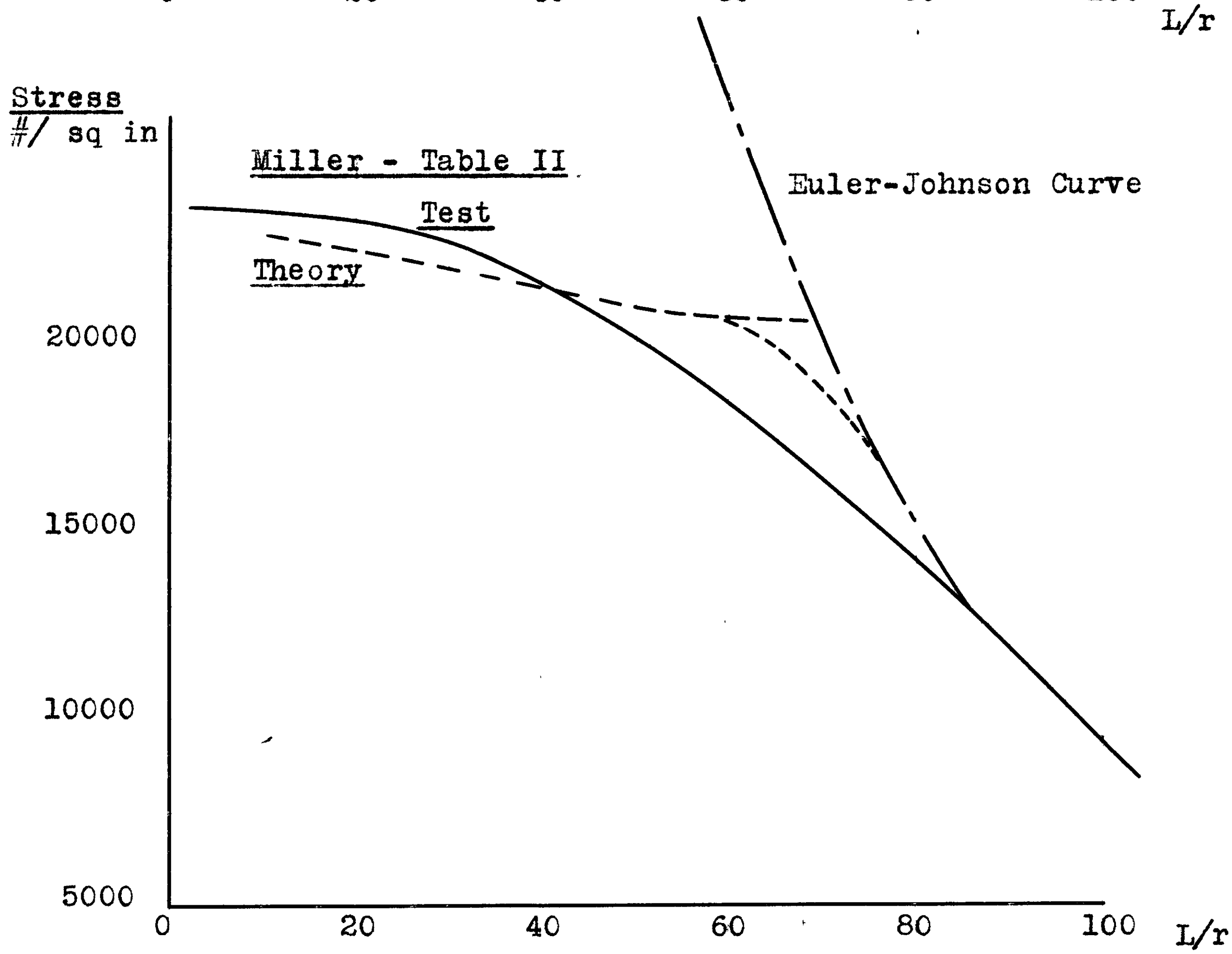
$E = 9\ 730\ 000$ $\sigma = 3/10$

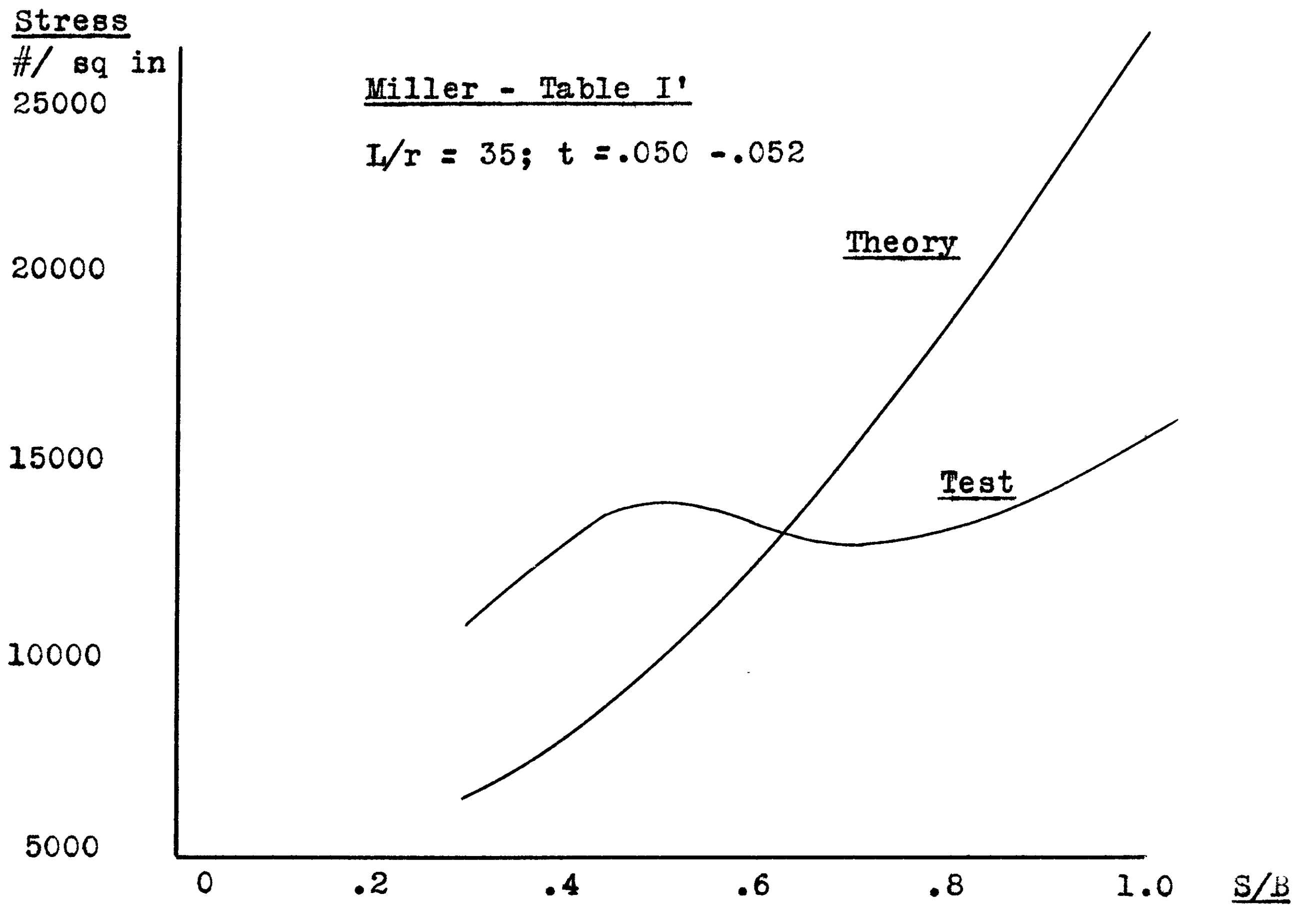
L/r	L	S	B	t	S/t	B/t	S/B	C
34.5	18.20	1.62	1.60	.062	26.1	24.8	1.01	212
34.96	18.25	1.60	1.95	.064	25	30.5	.82	233.5
34.9	18.08	1.60	2.56	.063	25.4	40.6	.625	222.5
34.56	17.8	1.62	3.25	.063	25.7	51.6	.50	222.5
34.8	17.43	1.61	3.90	.063	25.6	61.9	.413	222.5
34.5	17.05	1.62	4.50	.063	25.7	71.4	.36	222.5

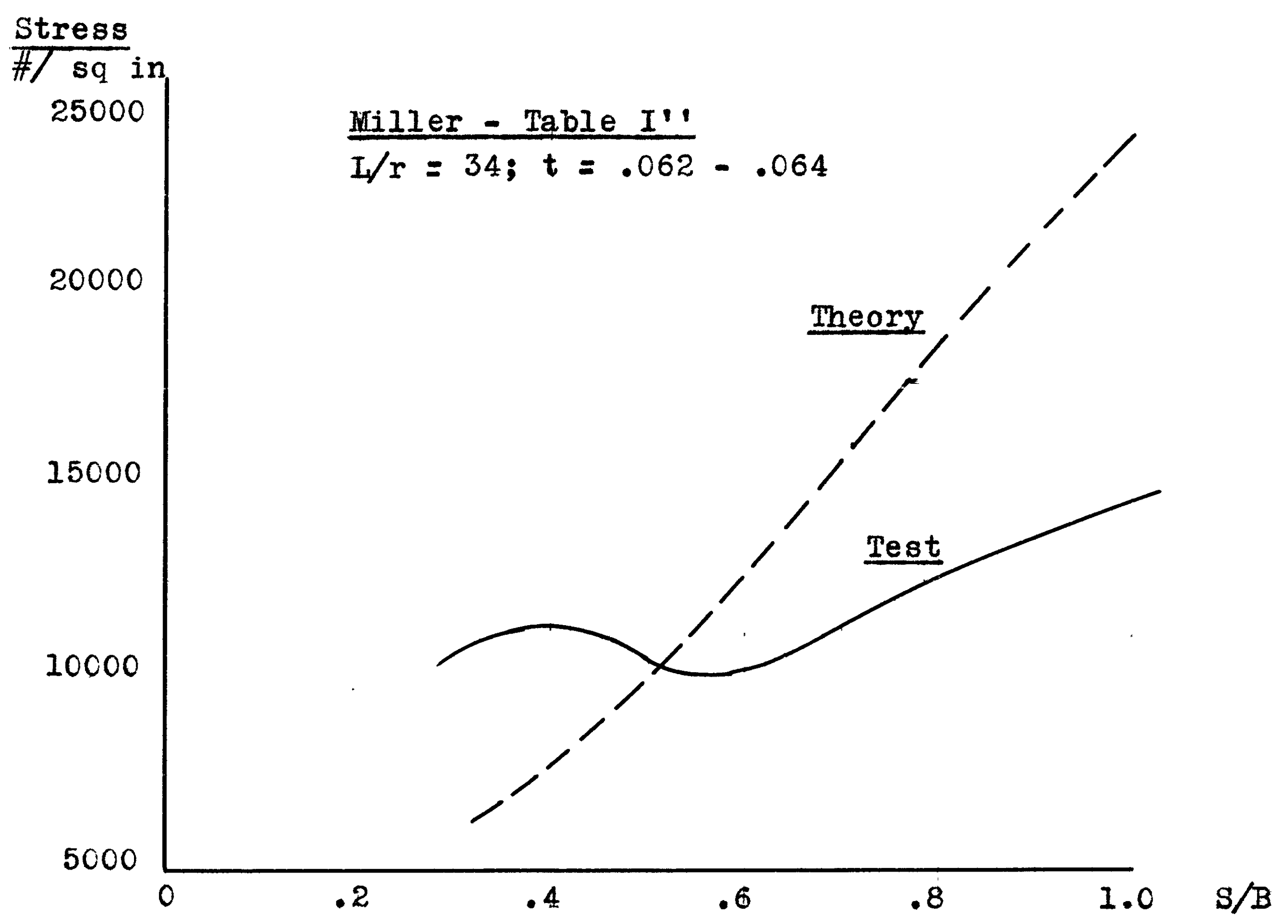
L/r	Calc. Load	Av. Test Load	Area	Calc. Stress	Test Stress
34.5	6350	3965	.2808	22600	14120
34.96	5980	4085	.3093	19860	13210
34.9	4628	3525	.3431	13500	10275
34.56	3874	4050	.3891	9970	10410
34.8	3432	5135	.4288	8000	11975
34.5	3130	5230	.4678	6590	11180

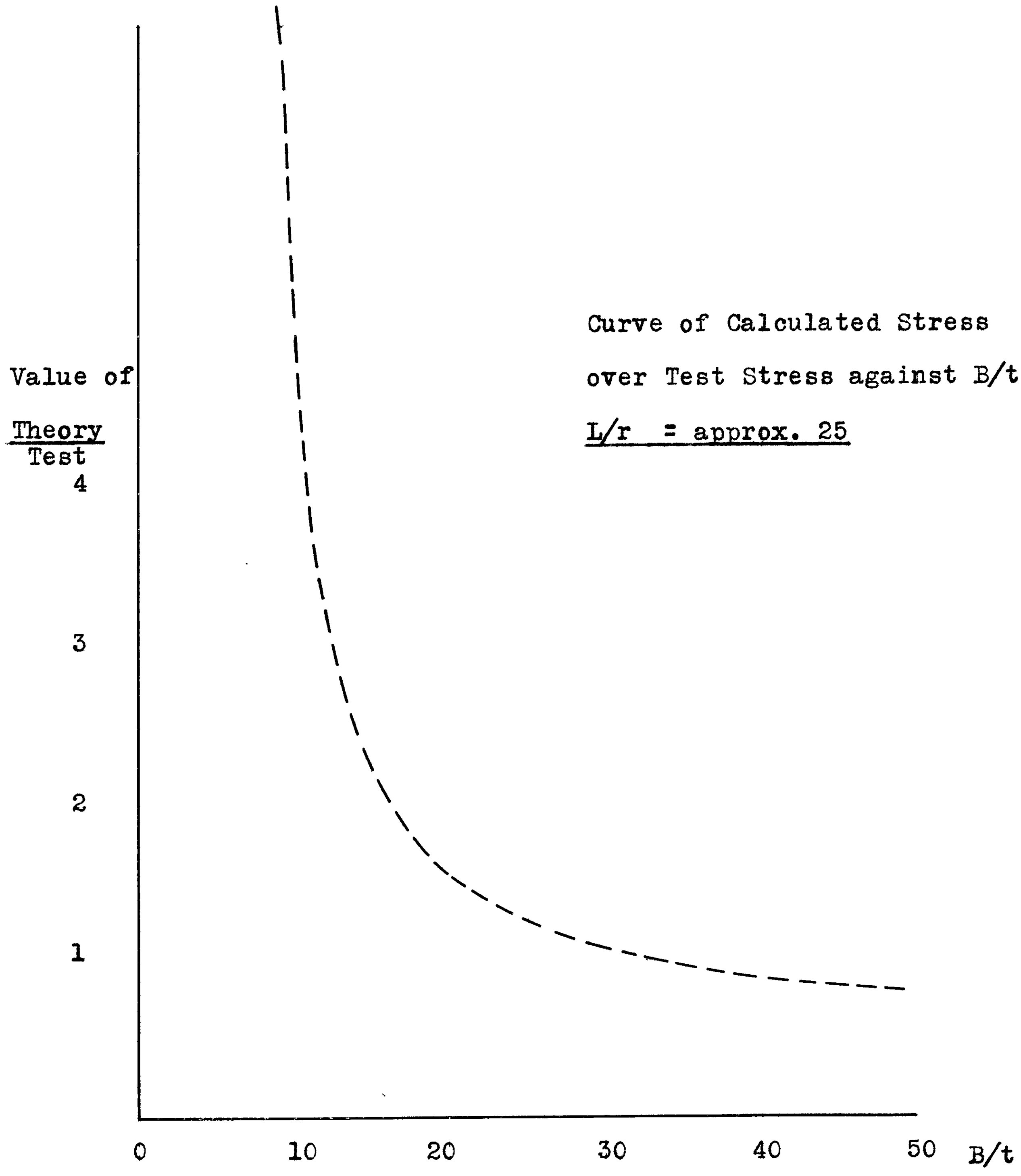

Stress
#/ sq in
40000
30000
20000
10000
0
0
20
40
60
80
100
L/r
Euler Curve
Test
Theory
Becker & Noveck - Mark A-1


Stress
#/ sq in
20000
15000
10000
5000
Miller - Table I
Euler-Johnson Curve
Test
Theory
0
20
40
60
80
100
L/r


Stress
#/ sq in
Miller - Table II
Euler-Johnson Curve
Test
Theory
20000
15000
10000
5000
0
20
40
60
80
100
L/r


Stress
#/ sq in
25000
20000
15000
10000
5000
Miller - Table I'
L/r = 35; t =.050 -.052
Theory
Test
0
.2
.4
.6
.8
1.0
S/B


Stress
#/ sq in
25000
20000
15000
10000
5000
Miller - Table I''
L/r = 34; t = .062 - .064
Theory
Test
0
.2
.4
.6
.8
1.0
S/B


Curve of Calculated Stress
over Test Stress against B/t
L/r = approx. 25
Value of
Theory
Test
4
3
2
1
0
10
20
30
40
50
B/t

DISCUSSION OF RESULTS AND CONCLUSIONS

In connection with the experimental data used in the preceeding tables, it may be mentioned that in most cases the values are the averages of several tests of presumably identical columns. In some cases the divergence between test values was considerable, the average in Miller's Table I being about 1500 #/ sq. inch. The Becker & Noveck tests were the averages of three runs on identical columns, and in most cases were in fairly closely agreement with each other.

The computed results, together with the accompanying curves, show that within certain limits, the theoretical formulas will give the critical stress with a fair degree of accuracy. The geometrical relations which have to be fulfilled by the dimensions of the section are indicated by the last two sets of curves, and it is evident that the conditions of the theory are most closely approximated when S/B is about ½, and B/t is about 30. If B/t decreases much below 25, the critical stress as given by the formulas is greatly overestimated.

Since many of the lighter types of columns sections fulfill these relations of back to sides and back to thickness, the theory may be applied to a fairly wide range.

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